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Final Exam for Complex Analysis **27/01/2025**

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Please write your name and student number on top of each sheet that you hand in.

The following standard notation is used throughout: For any complex number $\mathbf{w}$,
$\mathrm{Im}(\mathbf{w})$ = Imaginary part of $\mathbf{w}$, and $\overline{\mathbf{w}}$ = the conjugate of $\mathbf{w}$.
You get 10 points for participating. *Good luck!*

1. (15pts) Let $f(z)$ be an entire function with $\mathrm{Im}(f(z)) \geq 2$ for all $z \in \mathbb{C}$. Prove that f is constant.
   Suggestion: Consider the function $g(z) = e^{if(z)}$.

2. (15 pts) Let C denote the circle $|z| = 3$ oriented counterclockwise. Compute the integral
$$\int_C (\sin(z^2) + \overline{z})\, dz.$$

3. (15 pts) Consider the polynomial $p(z) = iz^7 + 5z^5 + 1$. Use Rouché's theorem to prove that all zeros of p are contained in the annulus $\frac{1}{2} < |z| < 3$.

4. (a) (10 pts) Determine the radius of convergence of the power series
$$h(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n + 1}$$
   (b) (10 pts) Let γ denote the unit circle $|z| = 1$ oriented counter clockwise. Find
$$\int_\gamma \frac{h(z)}{z^3}.$$

5. (a) (10 pts) Let C_R denote the half circle $\{z : |z| = R,\ \mathrm{Im}(z) \geq 0\}$. Prove that
$$\lim_{R\to\infty} \int_{C_R} \frac{z^2}{z^4+1}\, dz = 0$$
   (b) (15 pts) Using complex residues, compute the (real valued) integral
$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx.$$